Project Two

Analysis of a Projectile Fired from a Musket

March 28th, 2019

Abstract

The motion of a musket ball being shot will be subjected to various phenomenon such as the Coriolis force and air resistance. The ball has a greatly reduced range because of a quadratic air resistance. It was determined that the air resistance contributes more to the range compared to the Coriolis force alone.

1 Introduction

The projectile that was chosen to be fired was a musket ball from a Brown Bess rifle (Refshauge, 2015). Analysis on the projectile began under simple conditions and was gradually built up to a fairly realistic model. The latitude of UOIT is 43.9459° N and 78.8967° W. The starting position of the musket ball is located at the "Gate to the Future" sculpture on the University of Ontario Institute of Technology campus.

2 The Problem

An assumption was made to keep the problem manageable. To begin it is assumed that the musket ball will not rotate in the air. If the musket ball were to rotate then this problem would include concepts used in fluid dynamics as air flow around the object would further complicate the motion of the projectile. The musket ball's mass is 0.0324 kg and the angle it was fired from was 45 degrees.

2.1 Basic Projectile motion

First and foremost, a coordinate system must be defined. The one used to analyze motion of a projectile is shown in figure 1 below:

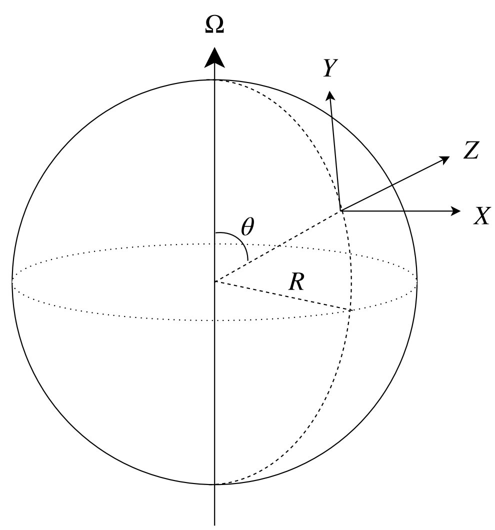

Figure 1: Coordinate system of a projectile on Earth.

where θ is the angle from the axis of rotation, R is the radius and Ω is the angular velocity of the earth. X points east, Y points north and Z points radially outwards.
A previous paper done by Kwok and Sands (2019) included a simple derivation of a projectile under the influence of gravity. When the new coordinate system is applied, the resulting equation becomes

$$x(\phi) = \frac{v_0^2 \sin 2\phi}{g} \tag{1}$$

where ϕ is the angle of launch relative to the ground. The projectile follows a fairly uninteresting path:

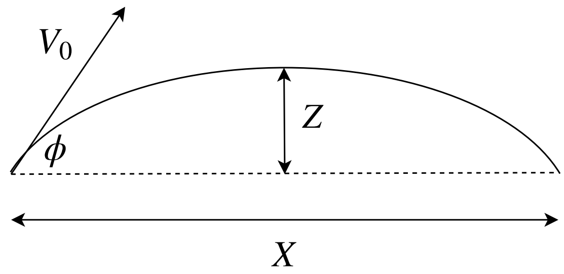

Figure 2: Simple projectile motion under the influence of gravity.

Although this model is a good introduction to projectile motion, a few other factors must be considered if a more realistic model is to be achieved.

2.2 Motion with Air resistance

Air resistance is not negligible in the motion of the musket ball, so it must be included. Now the motion of the ball must account for the drag force. Two terms make up the drag force, the linear term and the quadratic term. The linear term comes from the viscosity of the medium through which the object was moving through and the quadratic term accounts for the air that has been accelerated by the object. The equations for both forces are,

$$f_{lin} = bv \tag{2}$$

and,

$$f_{quad} = cv^2. \tag{3}$$

The b and c coefficients are,

$$b = \beta D$$

and,

$$c = \gamma D^2$$

where γ and β are dependent on the medium and D is the diameter of the object. Both of these drag forces are not always necessary and the technique to find out which force is more important is as follows,

$$\frac{f_{quad}}{f_{lin}} = \frac{\gamma D}{\beta} v$$

where $\gamma = 0.25N \cdot \dfrac{s^2}{m^4}$ and $\beta = 1.6 \cdot 10^{-4} N \cdot \dfrac{s}{m^2}$ (Taylor, 2005). Using the values $D = 0.016256$ m and muzzle velocity $v = 390.1\ m/s$, shows that the ratio is heavily in favour of the quadratic force. So the force is,

$$f = -cv^2\hat{v} = -cv\boldsymbol{v} = -c\sqrt{v_x^2 + v_y^2 + v_z^2}\ \boldsymbol{v}. \tag{4}$$

There will be 3 equations for each direction,

$$\ddot{x} = \frac{-c}{m}\sqrt{v_x^2 + v_y^2 + v_z^2}\ v_x, \tag{5}$$

$$\ddot{y} = \frac{-c}{m}\sqrt{v_x^2 + v_y^2 + v_z^2}\ v_y, \tag{6}$$

$$\ddot{z} = \frac{-c}{m}\sqrt{v_x^2 + v_y^2 + v_z^2}\, v_z - g. \tag{7}$$

Unfortunately, these equations are coupled together and are difficult to solve. So they will be solved numerically using Euler's method. Figures 3, 5, 4, 6 are what follows from Euler's method.

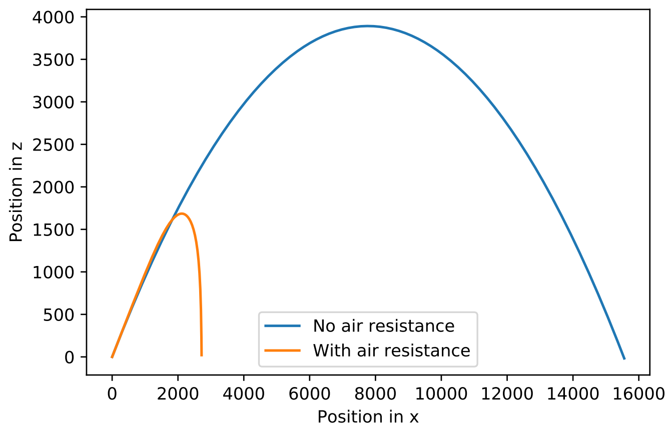

Figure 3: The effect of air resistance versus no air resistance on the musket ball.

2.3 The Centrifugal Force and the Coriolis Force

In a rotating reference frame, the equation of motion appears to be similar to that of Newton's Second Law, but with two additional terms:

$$m\ddot{r} = F + m(\Omega \times r) \times \Omega + 2m\dot{r} \times \Omega. \tag{8}$$

The first additional term is known as the centrifugal force

$$F_{cf} = m(\Omega \times r) \times \Omega. \tag{9}$$

This force points radially outward from the rotational axis and acts on objects that are at rest in the rotational frame. The second additional term is the Coriolis force,

$$F_{cor} = 2m\dot{r} \times \Omega. \tag{10}$$

This force acts on objects that are moving. The Coriolis force contains $\dot{r}$, meaning that it is proportional to the velocity of the object relative to the rotating frame. The object will trace out a curved path in flight as it is pushed along the cross product of $\dot{r} = v$ and Ω.

A projectile will experience the centrifugal and Coriolis force when it is launched a significant distance on earth. It should be noted that both of the forces mentioned here are considered fictitious forces, meaning that they don't actually exist in an inertial frame, but appear to in a rotating one (Taylor, 2005).

When the net force in the inertial frame is solely due to gravity, the first two terms on the right side of equation (8) can be combined to form

$$\ddot{r} = g_{eff} + 2\dot{r} \times \Omega \tag{11}$$

where a factor of m has been cancelled out and g_{eff} is the effective gravity that results from adding the centrifugal force to the force of gravity.

Currently, the origin is set at the centre of the Earth. Since equation (11) does not depend on the position r, the origin can be shifted to a more convenient location, say, the surface of the Earth at position R. $\dot{r}$ and Ω can now be written as

$$\dot{r} = (\dot{x}, \dot{y}, \dot{z}) \quad \text{and} \quad \Omega = (0, \Omega \sin\theta, \Omega \cos\theta) \tag{12}$$

which can easily be seen on figure 1.
After computing the cross product, equation (11) turns out to be

$$\ddot{r} = g_{eff} + 2(\dot{y}\Omega \cos\theta - \dot{z}\Omega \sin\theta, -\dot{x}\Omega \cos\theta, \dot{x}\Omega \sin\theta) \tag{13}$$

which result in the equations of motion:

$$\ddot{x} = 2\Omega(\dot{y} \cos\theta - \dot{z} \sin\theta), \tag{14}$$

$$\ddot{y} = -2\Omega\dot{x} \cos\theta, \tag{15}$$

$$\ddot{z} = -g + 2\Omega\dot{x} \sin\theta. \tag{16}$$

These three equations can be solved analytically using perturbation theory. A series of successive approximations can be made in which Ω starts at zero for the zeroth order approximation and becomes very small for the first order approximation.
The centrifugal force term contains a magnitude proportional to Ω^2 while the Coriolis force only has Ω. If Ω is already assumed to be small in the first order approximation, then Ω^2 will be even smaller. This makes the centrifugal force negligible as approximations go on, meaning that the Coriolis force will be the determining factor in the projectile's trajectory.

Although an analytical solution is possible, a numerical solution will provide a much more useful interpretation of the projectile's motion.
Once again, Euler's method was used to plot the motion of the the projectile. This time, the figure combines air resistance and inertial forces in various perspectives:

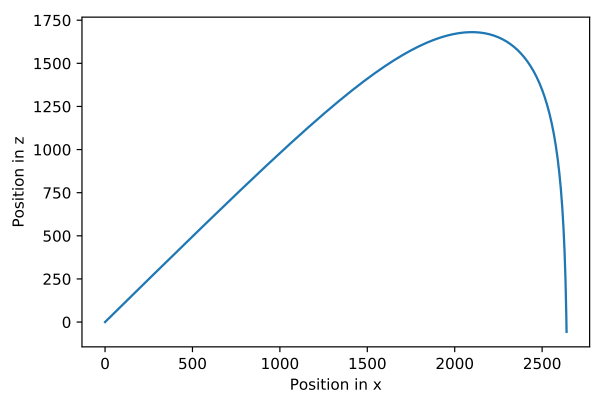

Figure 4: The effect of air resistance on the musket ball (with the Coriolis force).

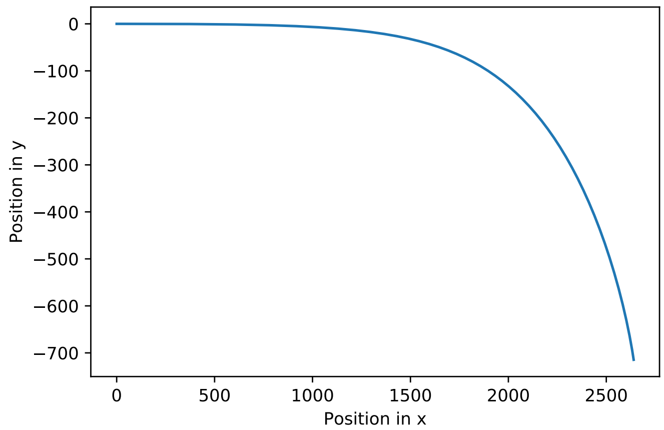

Figure 5: The effect of the Coriolis force on the musket ball.

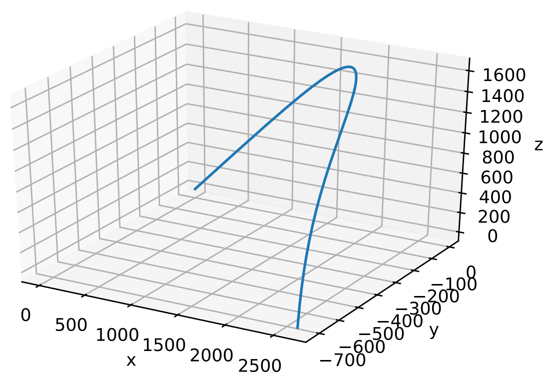

Figure 6: The effect of the Coriolis force on the musket ball

Figures 7 - 10 display errors for x positions and z positions.

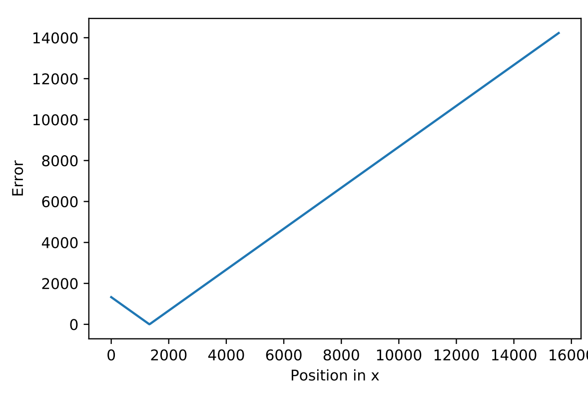

Figure 7: The error of the x position of no air resistance versus the effects of the Coriolis force and air resistance.

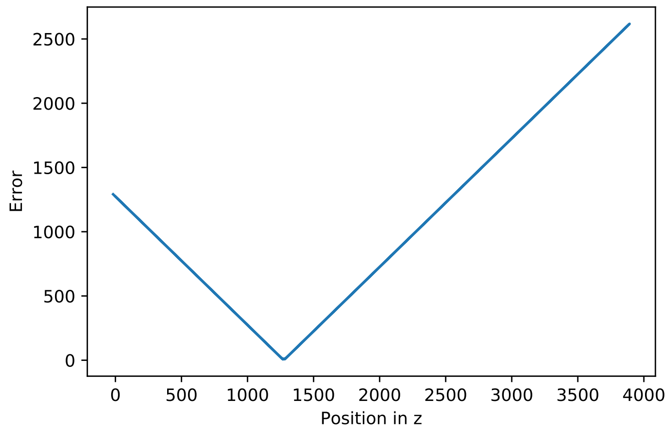

Figure 8: The error of the z position of no air resistance versus the effects of the Coriolis force and air resistance.

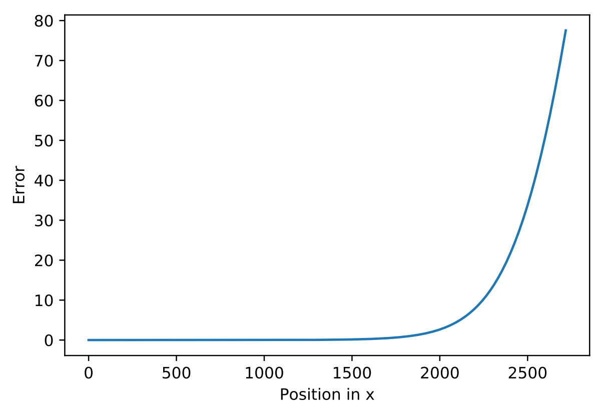

Figure 9: The error of the x position of air resistance versus the effects of the Coriolis force and air resistance.

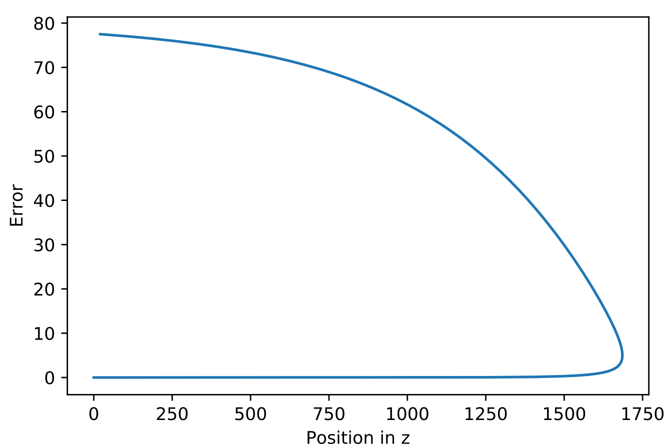

Figure 10: The error of the z position of air resistance versus the effects of the Coriolis force and air resistance.

3 Conclusion

Air resistance is a huge factor in how far the musket ball travels. According to the plots, the musket ball is heavily affected by the Coriolis force as well, but it seems unreasonable that it would travel 700 m south even if the musket ball travelled 2500 m. In figure 7, the error of the projectile's position in the x direction follows a linear trend upward after a slight decrease as the projectile flies through the air. This comes from the fact that the air resistance significantly affects the projectile's range. The error in z follows a similar trend in figure 8, but at a smaller scale. When the air resistance is compared to both the Coriolis force and the air resistance in figure 9 the error remains relatively low then exponentially increases for the x direction. The error in figure 10 is unexpected and may have arisen from a mistake in estimating the error.

References

Refshauge, W. F. (2015). A note on physical properties of musket fire. Journal of Conflict Archaeology, 10(3), 149-153. doi:10.1080/15740773.2016.1181854

Taylor, J. R. (2005). Classical mechanics. Sausalito, Calif: University Science Books.